arXiv:math-ph/0703039v1 13 Mar 2007

FERTILE THREE STATE HARD-CORE MODELS ON A CAYLEY TREE

U.A. Rozikov[1,2], **Sh.A. Shoyusupov**[1]

[1] *Institute of Mathematics, Tashkent, Uzbekistan.*

e-mail: rozikovu@yandex.ru, shokir79@mail.ru

[2] *School of Mathematical Sciences, GC University, Lahore, Pakistan.*

Abstract. We consider nearest-neighbor fertile hard-core models, with three states , on a homogeneous Cayley tree. It is known that there are four type of such models. We investigate all of them and describe translation-invariant and periodic hard-core Gibbs measures. Also we construct a continuum set of non-periodic Gibbs measures.

1 Introduction

A Cayley tree $T^k = (V, L)$ of order $k \geq 1$ is defined as an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each site. Here V is the set of sites and L is the set of edges. Fix a site x^0 (the origin) and set: $V_n = \{x \in V :$ dist $(x^0, x) \leq n\}$, $W_n = \{x \in V :$ dist $(x^0, x) = n\}$, where the distance between x, $y \in V$ is the number of edges in the shortest path $x \to y$.

We consider nearest-neighbor hard-core models, with three states , on a homogeneous Cayley tree. In these models one assigns, to each site x, values $\sigma(x) \in \{0, 1, 2\}$. Values $\sigma(x) = 1, 2$ mean that site x is 'occupied' and $\sigma(x) = 0$ that x is 'vacant'.

A configuration σ on the tree is a collection $\{\sigma(x), x \in V\}$ considered also as a function $V \to \{0, 1, 2\}$. In a similar fashion one defines a configuration in V_n and W_n.

In this paper we consider the fertile graphs (see [2], p.248) with three vertices $0, 1, 2$ (on the set of values $\sigma(x)$), with edges and loops as follows:

the *"wrench"*: $\{0, 1\}, \{0, 2\}$; loops at 0 and 1;

the *"wand"*: $\{0, 1\}, \{0, 2\}$; loops at 1 and 2;

the *"hinge"*: $\{0, 1\}, \{0, 2\}$; loops at 0, 1 and 2;

the *"pipe"*: $\{0, 1\}, \{1, 2\}$; loop at 0.

Denote $O = \{wrench, wand, hinge, pipe\}$.

Another graph which is non fertile is called sterile (see [2], p.247).

For $G \in O$ we call σ a $G-$admissible configuration (on the tree, in V_n or W_n) if $\{\sigma(x), \sigma(y)\}$ is an edge of G $\forall$ nearest-neighbor pair x, y (from V, V_n or W_n, respectively). Denote the set of $G-$admissible configurations by Ω^G ($\Omega^G_{V_n}$ and $\Omega^G_{W_n}$).

A set of activities (see [2]) for a graph G is a function $\lambda : G \to R_+$ from the vertices of G to the positive reals. The value λ_i of λ at a vertex $i \in \{0, 1, 2\}$ is called its "activity".

For a given G and λ we define Hamiltonian of the $(G-)$ hard core model as

$$H^\lambda_G(\sigma) = \begin{cases} \sum_{x \in V} \ln \lambda_{\sigma(x)}, & \text{if } \sigma \in \Omega^G, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

The hard-core model is interesting from the point of view of statistical mechanics, as well of combinatorics and the theory of neuron networks [3], [5].

Let $\mathbf{B}$ be the sigma-algebra generated by the cylinder subsets of Ω^G. Furthermore, $\forall$ n, $\mathbf{B}_{V_n}$ stands for the sub-algebra of $\mathbf{B}$ generated by events $\{\sigma\in\Omega^G\colon\ \sigma|_{V_n}=\sigma_n\}$ where $\sigma_n\colon\ x\in V_n\mapsto\sigma_n(x)$ is an admissible configuration in V_n and $\sigma|_{V_n}$ the restriction of σ on V_n.

Definition 1. A (three state) $G-$hard core Gibbs measure is a probability measure μ on $(\Omega^G,\mathbf{B})$ such that, $\forall$ n and $\sigma_n\in\Omega^G_{V_n}$:

$$\mu\left\{\sigma\in\Omega^G:\ \sigma|_{V_n}=\sigma_n\right\}=\int_{\Omega^G}\mu(d\omega)P_n(\sigma_n\,|\,\omega_{W_{n+1}}),\tag{1.2}$$

where

$$P_n(\sigma_n\,|\,\omega_{W_{n+1}})=\frac{\exp(-H^\lambda_G(\sigma_n))}{Z_n(\lambda;\omega|_{W_{n+1}})}\mathbf{1}\left(\sigma_n\vee\omega|_{W_{n+1}}\in\Omega^G_{V_{n+1}}\right).$$

Symbol $\vee$ means concatenation of configurations and $Z_n\left(\lambda;\omega|_{W_{n+1}}\right)$ is the partition function with the boundary condition $\omega|_{W_n}$:

$$Z_n\left(\lambda;\omega|_{W_{n+1}}\right)=\sum_{\widetilde{\sigma}_n\in\Omega^G_{V_n}}\exp(-H^\lambda_G(\sigma_n))\mathbf{1}\left(\widetilde{\sigma}_n\vee\omega|_{W_{n+1}}\in\Omega^G_{V_{n+1}}\right).\tag{1.3}$$

In [2] it was proven that (i) for every sterile graph G and any positive activity set on G there is a unique invariant Gibbs measure on Ω^G; (ii) for any fertile graph G there is a set of activities λ on G for which Ω^G has at least two simple, invariant Gibbs measures.

In this paper we shall consider the case $\lambda_0=1$, $\lambda_1=\lambda_2=\lambda>0$ and describe corresponding translation-invariant, periodic and some non-periodic Gibbs measures. In [7] these problems were solved for $G=$wrench case. So we shall consider cases hinge, pipe and wand. Our some results improve the analogical results of [7].

The paper is organized as follows. In section 2 we reduce our problems to solve a system of functional equations which depends on adjacency matrix of $G\in O$. Section 3 is devoted to translation-invariant Gibbs measures. Sections 4 and 5 are devoted to periodic and non-periodic Gibbs measures respectively. All sections contain some remarks which compare our results with known results.

2 System of functional equations

Write $x<y$ if the path from x^0 to y goes through y. Call vertex y a direct successor of x if $y>x$ and x,y are nearest neighbors. Denote by $S(x)$ the set of direct successors of x. Note that any vertex $x\neq x^0$ has k direct successors and x^0 has $k+1$.

For $\sigma_n\in\Omega^G_{V_n}$ we define : $\#\sigma_n=\sum\limits_{x\in V_n}\mathbf{1}(\sigma_n(x)\geq 1)$ (the number of occupied sites in σ_n).

Let $z: \ x \mapsto z_x = (z_{0,x}, z_{1,x}, z_{2,x}) \in \mathbf{R}^3_+$ be a vector-valued function on V. Given $n = 1, 2, \ldots$, and $\lambda > 0$ consider the probability distribution $\mu^{(n)}$ on $\Omega^G_{V_n}$ defined by

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \lambda^{\#\sigma_n} \prod_{x \in W_n} z_{\sigma(x),x}. \tag{2.1}$$

Here Z_n is the corresponding partition function:

$$Z_n = \sum_{\widetilde{\sigma}_n \in \Omega^G_{V_n}} \lambda^{\#\widetilde{\sigma}_n} \prod_{x \in W_n} z_{\widetilde{\sigma}(x),x}.$$

We say that the probability distributions $\mu^{(n)}$ are compatible if $\forall \ n \geq 1$ and $\sigma_{n-1} \in \Omega^G_{V_{n-1}}$:

$$\sum_{\omega_n \in \Omega^G_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \mathbf{1}(\sigma_{n-1} \vee \omega_n \in \Omega^G_{V_n}) = \mu^{(n-1)}(\sigma_{n-1}). \tag{2.2}$$

In this case there exists a unique probability measure μ on $(\Omega^G, \mathbf{B})$ such that, $\forall \ n$ and $\sigma_n \in \Omega^G_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$.

Definition 2. Measure μ defined by (2.1), (2.2) is called a $(G-)$hard core Gibbs measure with $\lambda > 0$, corresponding to function $z: \ x \in V \setminus \{x^0\} \mapsto z_x$. The set of such measures (for all possible choices of z) is denoted by $\mathcal{S}_G$.

For graph G denote by $L(G)$ the set of its edges and by $A \equiv A^G = (a_{ij})_{i,j=0,1,2}$ the adjacency matrix of G i.e.

$$a_{ij} \equiv a^G_{ij} = \begin{cases} 1, & \text{if } \ \{i,j\} \in L(G), \\ 0 & \text{otherwise.} \end{cases}$$

The following statement describes conditions on z_x guaranteeing compatibility of distributions $\mu^{(n)}$.

Theorem 1. *Probability distributions $\mu^{(n)}$, $n = 1, 2, \ldots$, in (2.1) are compatible iff for any $x \in V$ the following system of equations holds:*

$$z'_{1,x} = \lambda \prod_{y \in S(x)} \frac{a_{10} + a_{11} z'_{1,y} + a_{12} z'_{2,y}}{a_{00} + a_{01} z'_{1,y} + a_{02} z'_{2,y}},$$

$$z'_{2,x} = \lambda \prod_{y \in S(x)} \frac{a_{20} + a_{21} z'_{1,y} + a_{22} z'_{2,y}}{a_{00} + a_{01} z'_{1,y} + a_{02} z'_{2,y}}, \tag{2.3}$$

where $z'_{i,x} = \lambda z_{i,x} / z_{0,x}, \quad i = 1, 2.$

Proof. Left hand side of (2.2) can be written as:

$$\frac{1}{Z_n} \lambda^{\#\sigma_{n-1}} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \left(a_{\sigma_{n-1}(x)0} z_{0,y} + a_{\sigma_{n-1}(x)1} \lambda z_{1,y} + a_{\sigma_{n-1}(x)2} \lambda z_{2,y} \right). \tag{2.4}$$

Sufficiency. Suppose that (2.3) holds. It is equivalent to the representations

$$\prod_{y\in S(x)} (a_{i0}z_{0,y}+\lambda a_{i1}z_{1,y}+\lambda a_{i2}z_{2,y}) = a(x)z_{i,x}, \quad i=0,1,2. \tag{2.5}$$

for some function $a(x)>0$, $x\in V$. Setting $A_n=\prod_{x\in W_n} a(x)$ and substituting (2.1) into LHS of (2.2), we get (2.4) and by (2.5) we have

$$\frac{1}{Z_n}\lambda^{\#\sigma_{n-1}}\prod_{x\in W_{n-1}} z_{\sigma_{n-1}(x),x}a(x)=\frac{A_{n-1}}{Z_n}\lambda^{\#\sigma_{n-1}}\prod_{x\in W_{n-1}} z_{\sigma_{n-1}(x),x}.$$

We should have

$$\sum_{\sigma_{n-1}\in\Omega^G_{V_{n-1}}}\ \sum_{\omega_n\in\Omega^G_{W_n}} \mathbf{1}\left(\sigma_{n-1}\vee\omega_n\in\Omega^G_{V_n}\right)\mu^{(n)}(\sigma_{n-1}\vee\omega_n)=1.$$

hence $A_{n-1}/Z_n=1/Z_{n-1}$, and (2.2) holds.

Necessity. Suppose that (2.2) holds; we want to prove (2.3). Substituting (2.1) in (2.2) and using (2.4), we obtain that $\forall\ \sigma_{n-1}\in\Omega^G_{V_{n-1}}$:

$$\frac{1}{Z_n}\lambda^{\#\sigma_{n-1}}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\left(a_{\sigma_{n-1}(x)0}z_{0,y}+a_{\sigma_{n-1}(x)1}\lambda z_{1,y}+a_{\sigma_{n-1}(x)2}\lambda z_{2,y}\right)=\prod_{x\in W_{n-1}} z_{\sigma_{n-1}(x),x}.$$

From this equality follows

$$\frac{Z_{n-1}}{Z_n}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}(a_{i0}z_{0,y}+\lambda a_{i1}z_{1,y}+\lambda a_{i2}z_{2,y})=\prod_{x\in W_{n-1}} z_{i,x}, \quad i=0,1,2. \tag{2.6}$$

Denoting $z'_{i,x}=\lambda z_{i,x}/z_{0,x}$, $\ i=1,2$ from (2.6) we get (2.3). ∎

Remark 1. For G =wrench from Theorem 1 one gets Theorem 1 of [7].

Remark 2. One can similarly prove Theorem 1 for very general setting: $\sigma(x)$ takes values $0,1,...,q$; G is a fixed graph with $q\geq 1$ vertices; $\lambda: i\in G\to\lambda_i\in R_+$ is a given function. Then (2.3) has the form

$$z_{j,x}=\frac{\lambda_j}{\lambda_0}\prod_{y\in S(x)}\frac{a_{j0}+\sum_{i=1}^q a_{ji}z_{i,y}}{a_{00}+\sum_{i=1}^q a_{0i}z_{i,y}}, \quad j=1,2,...q.$$

However, the analysis of solutions of the equation is very difficult.

3 Translation-invariant Gibbs measures

We set in future $z_{0,x}\equiv 1$ and $z_{i,x}=z'_{i,x}>0$, $\ i=1,2$. Then $\forall$ function $x\in V\mapsto z_x=(z_{1,x},z_{2,x})$ satisfying

$$z_{i,x}=\lambda\prod_{y\in S(x)}\frac{a_{i0}+a_{i1}z_{1,y}+a_{i2}z_{2,y}}{a_{00}+a_{01}z_{1,y}+a_{02}z_{2,y}}, \quad i=1,2 \tag{3.1}$$

there exists a unique $G-$hard core Gibbs measure μ and vice versa. It is natural to begin with translation-invariant solutions where $z_x=z$ is constant.

3.1 Case hinge

In this case assuming $z_x = z$ we obtain from (3.1) the following system of equations:

$$\begin{cases} z_1 = \lambda\Big(\frac{1+z_1}{1+z_1+z_2}\Big)^k, \\ z_2 = \lambda\Big(\frac{1+z_2}{1+z_1+z_2}\Big)^k. \end{cases} \tag{3.2}$$

Subtracting from the first equation of system (3.2) the second one we get

$$(z_1 - z_2)\left[1 - \lambda\frac{(1+z_1)^{k-1} + ... + (1+z_2)^{k-1}}{(1+z_1+z_2)^k}\right] = 0.$$

Consequently, we have $z_1 = z_2$ and

$$(1+z_1+z_2)^k = \lambda((1+z_1)^{k-1} + ... + (1+z_2)^{k-1}), \tag{3.3}$$

if $z_1 \neq z_2$. For $z_1 = z_2 = z$ from system (3.2) we have

$$\lambda^{-1}z = f(z) = \Big(\frac{1+z}{1+2z}\Big)^k. \tag{3.4}$$

The function $f(z)$ is decreasing for $z > 0$ which implies that equation (3.4) has unique solution $z^* = z^*(k, \lambda)$ for any $\lambda > 0$.

If (3.3) is satisfied then we assume $k = 2$ and from (3.3) we have

$$1 + z_1 + z_2 = \frac{\lambda + \sqrt{\lambda^2 + 4\lambda}}{2}. \tag{3.5}$$

Using this equality from first equation of the system (3.2) we have for $k = 2$

$$z_1^{(1)} = \Big(\frac{1+\sqrt{1-4a^2}}{2a}\Big)^2, \quad z_1^{(2)} = \Big(\frac{1-\sqrt{1-4a^2}}{2a}\Big)^2, \tag{3.6}$$

if $\lambda > 9/4$ where $a = 2(\sqrt{\lambda}+\sqrt{\lambda+4})^{-1}$. Using the second equation we also have $z_2^{(1)}, z_2^{(2)} \in \{z_1^{(1)}, z_1^{(2)}\}$. Since $z_1 \neq z_2$ we conclude that $z_1 = z_1^{(1)}, z_2 = z_1^{(2)}$ and $z_1 = z_1^{(2)}, z_2 = z_1^{(1)}$. It is easy to check that these solutions satisfies the condition (3.5).

Thus if $k = 2$, $\lambda > \frac{9}{4}$ then the system (3.2) has three solutions (z^*, z^*), $(z_1^{(1)}, z_1^{(2)})$, $(z_1^{(2)}, z_1^{(1)})$, where z^* is the unique solution of (3.4) and $z_1^{(i)}$, $i = 1, 2$ is defined in (3.6). Note that $z_1^{(1)} = \frac{1}{z_2^{(2)}}$. Consequently by Theorem 1 we get the following

Theorem 2. *If $k = 2$ then for the hinge case*
1) for $\lambda \leq \frac{9}{4}$ there exists unique hard-core translation-invariant Gibbs measure μ_0;
2) for $\lambda > \frac{9}{4}$ there are at least three hard-core translation-invariant Gibbs measures μ_i, $i = 0, 1, 2$.

Remark 3. 1) Note that the idea of analysis of solutions (3.2) is taken from [7].
2) The value $\lambda = \lambda_{\mathrm{cr}} = \frac{9}{4}$ is exactly the critical value for $k = 2$. Clearly $\lambda_{\mathrm{cr}} < 4 = \lambda_{\mathrm{cr}}^{\mathrm{HC}}$ for

$k=2$. Here $\lambda_{\rm cr}^{\rm HC}=\frac{1}{k-1}(\frac{k}{k-1})^k$ is the critical value for two state hard-core model [10].

Proposition 1. *If $z_x=(z_{1,x},z_{2,x})$ is a solution of (3.1) in the case hinge then $z_i^-\leq z_{i,x}\leq z_i^+,\quad$ for any $\quad i=1,2,\quad x\in V$, where $(z_1^-,z_1^+,z_2^-,z_2^+)$ is a solution of*

$$\begin{cases} z_1^- = \lambda\Big(\frac{1+z_1^-}{1+z_1^-+z_2^+}\Big)^k, \\ z_1^+ = \lambda\Big(\frac{1+z_1^+}{1+z_1^++z_2^-}\Big)^k, \\ z_2^- = \lambda\Big(\frac{1+z_2^-}{1+z_1^++z_2^-}\Big)^k, \\ z_2^+ = \lambda\Big(\frac{1+z_2^+}{1+z_1^-+z_2^+}\Big)^k. \end{cases} \tag{3.7}$$

Proof. Is very similar to proof of Proposition 5 [7].

Proposition 2. *If $z=(z_1^-,z_1^+,z_2^-,z_2^+)$ a solution of (3.7) then $z_1^-=z_1^+$ iff $z_2^-=z_2^+$.*

Proof. See [7], Proposition 6.

Corollary 1. If the system (3.7) has unique solution then system (3.1) also has unique solution. Moreover this solution is $z_x=(z_1^*,z_2^*),\quad x\in V$ where (z_1^*,z_2^*) is the unique solution of (3.2).

Now we shall find exact values of $z_i^-,z_i^+,i=1,2$ for $k=2$.

Consider the system consisting of the first and the last equations of (3.7):

$$\begin{cases} z_1^- = \lambda\Big(\frac{1+z_1^-}{1+z_1^-+z_2^+}\Big)^k, \\ z_2^+ = \lambda\Big(\frac{1+z_2^+}{1+z_1^-+z_2^+}\Big)^k. \end{cases} \tag{3.8}$$

If $z_1^-=z_2^+$ then this system has unique solution. In case $z_1^-\neq z_2^+$ we get

$$(1+z_1^-+z_2^+)^k=\lambda((1+z_1^-)^{k-1}+...+(1+z_2^+)^{k-1}). \tag{3.9}$$

If $k=2$ then from (3.9) we have

$$1+z_1^-+z_2^+=\frac{\lambda+\sqrt{\lambda^2+4\lambda}}{2}. \tag{3.10}$$

Using this equality from first equation of the system (3.8) we have for $k=2$

$$(z_1^-)^{(1)}=\Big(\frac{1+\sqrt{1-4a^2}}{2a}\Big)^2,\quad (z_1^-)^{(2)}=\Big(\frac{1-\sqrt{1-4a^2}}{2a}\Big)^2, \tag{3.11}$$

if $\lambda>\frac{9}{4}$ where $a=2(\sqrt{\lambda}+\sqrt{\lambda+4})^{-1}$. Using $(z_1^-)^{(i)},i=1,2$ and (3.10) we get $(z_2^+)^{(1)}=\Big(\frac{1-\sqrt{1-4a^2}}{2a}\Big)^2$, $(z_2^+)^{(2)}=\Big(\frac{1+\sqrt{1-4a^2}}{2a}\Big)^2$. Similarly from the second and third equality of (3.7) we get

$$z_1^+\in M=\Big\{\Big(\frac{1+\sqrt{1-4a^2}}{2a}\Big)^2,\Big(\frac{1-\sqrt{1-4a^2}}{2a}\Big)^2\Big\},\quad z_2^-\in M.$$

Note that $(z_i^{\pm})^{(1)} = \frac{1}{(z_i^{\mp})^{(2)}}, \quad i = 1, 2.$

Thus we proved

Proposition 3. *If $k = 2$ then for the case hinge*

1) for $\lambda \leq \frac{9}{4}$ system (3.7) has unique solution z^*;

2) for $\lambda > \frac{9}{4}$ system (3.7) has three solutions $z_1^* = (z^-, \frac{1}{z^-}, z^-, \frac{1}{z^-})$, $z_2^* = (\frac{1}{z^-}, \frac{1}{z^-}, z^-, z^-)$, $z_3^* = (z^-, z^-, \frac{1}{z^-}, \frac{1}{z^-})$ where $z^- = \left(\frac{1-\sqrt{1-4a^2}}{2a}\right)^2$.

Note that for $\lambda > \frac{9}{4}$ we have $0 < a < \frac{1}{2}$ and $z^- < 1$.

Corollary 2. *If $k = 2, \lambda > \frac{9}{4}$ then for any solution of (3.1) (case hinge) we have $z^- \leq z_{i,x} \leq \frac{1}{z^-}$, $i = 1, 2$.*

Remark 4. To get exact solutions of (3.7) for $k = 2$ we used independence of first and last equations of (3.7) from the second and third ones. But there is not such an independence for the pipe and wrench case. Thus an analogue of Corollary 2 is not clear for these cases.

3.2 Case wand.

In this case from (3.1) for $z_x = z$ we have

$$\begin{cases} z_1 = \lambda\left(\frac{1+z_1}{z_1+z_2}\right)^k, \\ z_2 = \lambda\left(\frac{1+z_2}{z_1+z_2}\right)^k. \end{cases} \tag{3.12}$$

This case is very similar to the case hinge and one can prove that if $k = 2, \quad \lambda > 1$ then the system (3.12) has three solutions given by similar formulas of case hinge just replacing a with $a = 2(\sqrt{\lambda} + \sqrt{\lambda + 8})^{-1}$.

Thus one can formulate an analogue of Theorem 2 with $\lambda_{cr} = 1$. But we have not analogues of Propositions 1-3 for the case wand.

3.3 Case pipe

In this case from (3.1) for $z_x = z$ we have

$$\begin{cases} z_1 = \lambda\left(\frac{1+z_2}{1+z_1}\right)^k, \\ z_2 = \lambda\left(\frac{z_1}{1+z_1}\right)^k. \end{cases} \tag{3.13}$$

From this we get $(x = z_2)$

$$\lambda^{-1}x = f(x) = \Big(\frac{\sqrt[k+1]{x(1+x)^k}}{1 + \sqrt[k+1]{x(1+x)^k}}\Big)^k. \tag{3.14}$$

We have

$$f'(x) = \frac{k}{k+1} \cdot \frac{(k+1)x+1}{x(x+1)} \cdot \frac{(\sqrt[k+1]{x(1+x)^k})^k}{(1+\sqrt[k+1]{x(1+x)^k})^{k+1}} > 0.$$

Note that the equation (3.14) has at least one positive solution, since f is increasing and $f(0) = 0, f(+\infty) = 1$. It is easy to see that equation (3.14) has more than one positive solution if and only if there is more than positive solution to $xf'(x) = f(x)$, which is the same as

$$(k^2-1)x = \varphi(x) = (k+1)(1+x)\sqrt[k+1]{x(1+x)^k} + 1. \tag{3.15}$$

Repeating this argument one can see that (3.15) has more than one solution if and only if such is $x\varphi'(x) = \varphi(x)$. This equation has the form

$$(k+1)x = \psi(x) = \frac{1}{\sqrt[k+1]{x(1+x)^k}} + k. \tag{3.16}$$

Since the function $\psi(x)$ is decreasing the equation (3.16) has unique solution. Consequently the system (3.13) has unique solution.

Thus we have proved

Theorem 3. *For the case pipe $\forall \lambda > 0$, $\forall k \geq 1$, the translation-invariant pipe-hard core Gibbs measure is unique.*

Remark 5. For $k = 2$ this theorem was proved in [7].

For the case pipe one can prove the following propositions which are analogues of Propositions 1 and 2.

Proposition 4. *If $z_x = (z_{1,x}, z_{2,x})$ is a solution of (3.1) in the case pipe then $z_i^- \leq z_{i,x} \leq z_i^+$, for any $i = 1, 2$, $x \in V$, where $(z_1^-, z_1^+, z_2^-, z_2^+)$ is a solution of*

$$\begin{cases} z_1^- = \lambda\left(\frac{1+z_2^-}{1+z_1^+}\right)^k, \\ z_1^+ = \lambda\left(\frac{1+z_2^+}{1+z_1^-}\right)^k, \\ z_2^- = \lambda\left(\frac{z_1^-}{1+z_1^-}\right)^k, \\ z_2^+ = \lambda\left(\frac{z_1^+}{1+z_1^+}\right)^k. \end{cases} \tag{3.17}$$

Proposition 5. *If $z = (z_1^-, z_1^+, z_2^-, z_2^+)$ a solution of (3.17) then $z_1^- = z_1^+$ iff $z_2^- = z_2^+$.*

Remark 6. 1) For the case pipe we have not an analogue of Proposition 3 and Corollary 2 since in this case there is no an independence (mentioned in Remark 4) between equations of (3.17).

2) Next two sections are devoted to description of periodic and some non-periodic Gibbs measures for the case hinge. Results of these sections can be similarly proved for case pipe. But for the case wand one needs to prove an analogue of Proposition 2.

4 Description of periodic Gibbs measures: case hinge

For the case hinge we write (3.1) in the following form

$$h_{1,x} = \ln \lambda + \sum_{y \in S(x)} \ln \frac{1 + \exp(h_{1,y})}{1 + \exp(h_{1,y}) + \exp(h_{2,y})},$$

$$h_{2,x} = \ln \lambda + \sum_{y \in S(x)} \ln \frac{1 + \exp(h_{2,y})}{1 + \exp(h_{1,y}) + \exp(h_{2,y})}, \tag{4.1}$$

where $h_{i,x} = \ln z_{i,x}, \quad i = 1, 2$. In this section we study periodic solutions of system (4.1).

Note that (see [4]) there exists a one-to-one correspondence between the set V of vertexes of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups of the second order with generators $a_1, a_2, ..., a_{k+1}$.

Definition 3. Let H_0 be a subgroup of G_k. We say that a collection $h = \{h_x = (h_{1,x}, h_{2,x}) : x \in G_k\}$ is H_0-*periodic* if $h_{i,yx} = h_{i,x}$ for all $i = 1, 2$, $x \in G_k$ and $y \in H_0$.

Definition 4. A Gibbs measure is called H_0-*periodic* if it corresponds to an H_0-periodic collection h.

Observe that a translation-invariant Gibbs measure is G_k-periodic.

Define function $h = (h_1, h_2) \mapsto F(h) = (F_1(h), F_2(h))$ where

$$F_1(h) = \ln \frac{1 + \exp(h_1)}{1 + \exp(h_1) + \exp(h_2)}, \quad F_2(h) = \ln \frac{1 + \exp(h_2)}{1 + \exp(h_1) + \exp(h_2)}. \tag{4.2}$$

Proposition 6. *$F(h) = F(l)$ if and only if $h = l$.*

Proof. *Necessity.* Let $F(h) = F(l)$ then $F_1(h) = F_1(l)$, $F_2(h) = F_2(l)$, where $h = (h_1, h_2)$, $l = (l_1, l_2)$. From this equalities we obtain

$$\begin{cases} -t_2(z_1 - t_1) + (1 + t_1)(z_2 - t_2) = 0, \\ (1 + t_2)(z_1 - t_1) - t_1(z_2 - t_2) = 0, \end{cases} \tag{4.3}$$

where $z_i = \exp(h_i), \quad t_i = \exp(l_i), \quad i = 1, 2$. Note that determinant of system (4.3) is negative, i.e. $\Delta = -(1 + t_1 + t_2) < 0$. Therefore, (4.3) has unique solution $z_i = t_i, \quad i = 1, 2$.

Sufficiency. Straightforward.

Let G_k^2 be the subgroup in G_k consisting of all words of even length. Clearly, G_k^2 is a subgroup of index 2. H_0 be a normal subgroup of finite index in G_k. We put $I(H_0) = H_0 \cap \{a_1, ..., a_{k+1}\}$, where $a_i, \quad i = 1, ..., k+1$ are generators of G_k.

Using a similar argument of [7] one can prove following theorems.

Theorem 4. *For any normal subgroup of finite index each H_0- periodic Gibbs measure of hinge-hard core model is either translation-invariant or G_k^2-periodic.*

Theorem 5. *If $I(H_0) \neq \emptyset$ then each H_0- periodic Gibbs measure is translation-invariant.*

Theorems 4 and 5 reduce the problem of describing H_0- periodic Gibbs measure with $I(H_0)\neq\emptyset$ to describing the fixed points of the map $h=(h_1,h_2)\to(\ln\lambda,\ln\lambda)+kF(h)$, which describes translation-invariant Gibbs measures. If $I(H_0)=\emptyset$, this problem is reduced to describing the solutions of the system:

$$\begin{cases} h=(\ln\lambda,\ln\lambda)+kF(l), \\ l=(\ln\lambda,\ln\lambda)+kF(h). \end{cases} \tag{4.4}$$

Note that system (4.4) describes of periodic measures with period two, precisely, G_k^2- periodic measures.

Recall $z_i=\exp(h_i),\quad t_i=\exp(l_i),\quad i=1,2$. Then from (4.4) we get

$$\begin{cases} z_1=\lambda\left(\frac{1+t_1}{1+t_1+t_2}\right)^k, \\ z_2=\lambda\left(\frac{1+t_2}{1+t_1+t_2}\right)^k, \\ t_1=\lambda\left(\frac{1+z_1}{1+z_1+z_2}\right)^k, \\ t_2=\lambda\left(\frac{1+z_2}{1+z_1+z_2}\right)^k. \end{cases} \tag{4.5}$$

The analysis of solutions to system (4.5) is rather tricky. In a particular case we shall reduce the system (4.5) to equation $\gamma(\gamma(x))=x$ for some function γ and will apply the following lemma.

Lemma 1. (See [6]) *Let $f:[0,1]\to[0,1]$ be a continuous function with a fixed point $\xi\in(0,1)$. Assume that f is differentiable at ξ and that $f'(\xi)<-1$. Then there exist x_0, x_1, $0\leq x_0<\xi<x_1\leq 1$, such that $f(x_0)=x_1$ and $f(x_1)=x_0$.*

If $z_1=z_2=z$, $t_1=t_2=t$ then (4.5) reduces to following system

$$\begin{cases} z=\lambda\left(\frac{1+t}{1+2t}\right)^k, \\ t=\lambda\left(\frac{1+z}{1+2z}\right)^k. \end{cases} \tag{4.6}$$

Denote $\gamma(x)=\lambda\left(\frac{1+x}{1+2x}\right)^k$. Then from (4.6) we have

$$z=\gamma(t),\quad t=\gamma(z). \tag{4.7}$$

Note that the equation $x=\gamma(x)$ has unique solution $x^*=x^*(k,\lambda)$, for any $k\geq 1$ and $\lambda>0$.

Theorem 6. *For $k\geq 6$ and*

$$\lambda\in\left\{\lambda:\frac{k-3-\sqrt{(k-3)^2-8}}{4}<x^*<\frac{k-3+\sqrt{(k-3)^2-8}}{4}\right\} \tag{4.8}$$

there are three G_k^2-periodic measures μ_0, μ_*, μ_1,. Which corresponds to three solutions (x_0, x_1), (x_*, x_*), (x_1, x_0) of (4.7).

Proof. Note that function $\gamma(x)$ is decreasing for any $x > 0$. By Lemma 1, if x^* satisfies

$$\begin{cases} \gamma(x^*) = x^*, \\ \gamma'(x^*) < -1, \end{cases} \tag{4.9}$$

then (4.6) has two solutions. From (4.9) it follows that

$$2(x^*)^2 + (3-k)x^* + 1 < 0. \tag{4.10}$$

Solving this inequality we get $k \geq 6$ and (4.8). ∎

Remark 7. Theorem 6 gives more applicable conditions (for the case hinge) than Theorem 4 of [7] (for case wrench).

5 Non-Periodic Gibbs measures: case hinge

As follows from general results of [12,13], if a periodic Gibbs measure is non-unique then there exist at least countable many non-periodic Gibbs measures. This made more precise in theorem 8 below (for case hinge).

We will show that system of equations (4.1) admit uncountably many non-translational-invariant solutions. Take an arbitrary infinite path $\pi = \{x^0 = x_0 < x_1 < x_2 < ...\}$ on the Cayley tree starting at the origin $x_0 = x^0$. Establish a 1-1 correspondence between such paths and real numbers $t \in [0,1]$ ([8], [9]). Write $\pi = \pi(t)$ when it is desirable to stress the dependence upon t. Map path π to a function $h^\pi : x \in V \mapsto h_x^\pi$ satisfying (4.1). Note that π splits Cayley tree T^k into two subgraphs T_1^k and T_2^k.

For $k = 2, \lambda > \frac{9}{4}$ the function h^π is defined by

$$h_x^\pi = \begin{cases} \ln z^-, & \text{if } x \in T_1^k, \\ -\ln z^-, & \text{if } x \in T_2^k, \end{cases} \tag{5.1}$$

where $z^- = \left(\frac{1-\sqrt{1-4a^2}}{2a}\right)^2$, (see Proposition 3).

Let $h \mapsto F(h)$ be defined by (4.2).

Proposition 6. *For $k = 2$, $\lambda > \frac{9}{4}$ and any $h = (h_1, h_2) \in [\ln z^-; -\ln z^-]^2$ (recall $z^- < 1$) the following inequalities hold:*

a)

$$\left|\frac{\partial F_1}{\partial h_1}\right| \leq \frac{1}{(\sqrt{z^-+1}+\sqrt{z^-})^2}; \quad \left|\frac{\partial F_2}{\partial h_2}\right| \leq \frac{1}{(\sqrt{z^-+1}+\sqrt{z^-})^2};$$

$$\left|\frac{\partial F_1}{\partial h_2}\right| \leq \frac{1}{1+z^-+(z^-)^2}; \quad \left|\frac{\partial F_2}{\partial h_1}\right| \leq \frac{1}{1+z^-+(z^-)^2};$$

b)
$$\|F(h)-F(l)\| \leq \frac{2}{1+z^-+(z^-)^2}\|h-l\|.$$

Proof. a) Using lemma 9 of [11] and $\ln z^- \leq h_i \leq -\ln z^-, \quad i=1,2$ we have
$$\left|\frac{\partial F_1}{\partial h_1}\right| \leq \frac{\exp(h_2)}{(\sqrt{\exp(h_2)+1}+1)^2} = \psi(h_2).$$
The function $\psi(x)$ is increasing, therefore
$$\left|\frac{\partial F_1}{\partial h_1}\right| \leq \psi(-\ln z^-) = \frac{1}{(\sqrt{z^-+1}+\sqrt{z^-})^2}.$$
The proof for $\left|\frac{\partial F_2}{\partial h_2}\right|$ is similar.

Now consider
$$\left|\frac{\partial F_1}{\partial h_2}\right| = \frac{\exp(h_2)}{1+\exp(h_1)+\exp(h_2)} = \varphi(h_1,h_2).$$
Since $\varphi^{'}_{h_2} > 0$ and $\varphi^{'}_{h_1} < 0$ we have
$$\left|\frac{\partial F_1}{\partial h_2}\right| \leq \max \varphi(h_1,h_2) = \varphi(\ln z^-, -\ln z^-) = \frac{1}{1+z^-+(z^-)^2}$$
Similarly one can show that
$$\left|\frac{\partial F_2}{\partial h_1}\right| \leq \frac{1}{1+z^-+(z^-)^2}.$$

b) For $z^- < 1$ it is easy to see that $\frac{1}{(\sqrt{z^-+1}+\sqrt{z^-})^2} < \frac{1}{1+z^-+(z^-)^2}$. Using this inequality we obtain
$$\|F(h)-F(l)\| = \max_{i=1,2}\{|F_i(h)-F_i(l)|\} \leq$$
$$\max_{i=1,2}\{|(F_i)'_{h_1}||h_1-l_1| + |(F_i)'_{h_2}||h_2-l_2|\} \leq \frac{2}{1+z^-+(z^-)^2}\|h-l\|.$$
This completes the proof.

If $\frac{2}{1+z^-+(z^-)^2} < 1$ i.e. $z^- > \frac{\sqrt{5}-1}{2}$ then with the help of Proposition 6 it is easy to prove the following Theorem 7, similar to Theorem 3 of [9]:

Theorem 7. *If $k=2$, and λ is such that $\frac{\sqrt{5}-1}{2} < z^- < 1$ then for any infinite path π there exists a unique function h^π satisfying* (4.1) *and* (5.1).

In the standard way (see [4], [9], [12]) one can prove that functions $h^{\pi(t)}$ are different for different $t \in [0;1]$. Now let $\mu(t)$ denote the Gibbs measure corresponding to function $h^{\pi(t)}$, $t \in [0;1]$. Similarly to theorem 3.2 from [1], we can prove the following:

Theorem 8. *If conditions of Theorem 7 are satisfied then for any $t \in [0;1]$, there exists a unique hinge-hard core Gibbs measure $\mu(t)$. Moreover, the Gibbs measures μ_1, μ_2 (see Theorem 2) are specified as* $\mu(0) = \mu_1$ and $\mu(1) = \mu_2$.

Because measures $\mu(t)$ are different for different $t \in [0, 1]$ we obtain a continuum of distinct Gibbs measures which are non-periodic.

Acknowledgments. A part of this work was done within the scheme of Junior Associate at the ICTP, Trieste, Italy and the first author (UAR) thanks ICTP for providing financial support and all facilities (in May - August 2006). He also thanks the IHES, Bures-sur-Yvette, France for support and kind hospitality (in October - December 2006) and the SMS, Lahore, Pakistan where final part of this work was done.

References

1. Bleher P.M. and Ganikhodjaev N.N., On pure phases of the Ising model on the Bethe lattice. *Theor. Probab. Appl.* **35** (1990), 216-227.

2. Brightwell G. and Winkler P., Graph homomorphisms and phase transitions. *J. Combin. Theor.* Series B, **77**, (1999), 221-262.

3. Brightwell G., Häggström O. and Winkler P., Non monotonic behavior in hard-core and Widom-Rowlinson models. *J. Statist. Phys.*, **94**, (1999), 415-435.

4. Ganikhodjaev N. N. and Rozikov U. A., Description of periodic extreme Gibbs measures of some lattice models on the Cayley tree. *Theor. Math. Phys.***111** (1997), 480-486.

5. Georgii H.-O., Gibbs measures and phase transitions. *De Gruyter studies in math.* **9**, Berlin, New-York, 1988.

6. Kesten H., Quadratic transformations: A model for population growth. I. *Adv. Appl. Prob.* **2**, (1970), 1-82.

7. Martin J., Rozikov U.A., Suhov Yu.M., A Three state Hard-Core model on a Cayley tree, *J. Nonlinear Math. Phys.*, **12:3**, (2005), 432-448.

8. Rozikov U. A., Description of limiting Gibbs measures for $\lambda-$ models on the Bethe lattice, *Siberian Math. Journ.* **39** (1998), 427-435.

9. Rozikov U. A., Description uncountable number of Gibbs measures for inhomogeneous Ising model. *Theor. Math. Phys.***118** (1999), 95-104.

10. Rozikov U.A., Suhov Yu.M., A hard-core model on a Cayley tree: an example of a loss network, *Queueing Syst.*, **46**, (2004), 197-212.

11. Rozikov U. A., Shoyusupov Sh. A., Gibbs measures for the SOS model with four states on a Cayley tree. *Theor. Math. Phys.***149:1** (2006), 1312-1323.

12. Zachary S., Countable state space Markov random fields and Markov chains on trees. *Ann. Prob.* **11** (1983), 894–903.

13. Zachary S., Bounded, attractive and repulsive Markov specifications on trees and on the one-dimensional lattice. *Stochastic Process. Appl.* **20** (1985), 247–256.